Lefschetz Numbers and Geometry of Operators in W*-modules

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1 Introduction

The main goal of the present paper is to generalize the results of [18, 19] in the following way: To be able to define $K_0(A) \otimes \mathbf{C}$-valued Lefschetz numbers of the first type of an endomorphism V on a C*-elliptic complex one usually assumes that $V = T_g$ for some representation T_g of a compact group G on the C*-elliptic complex. We try to refuse this restriction in the present paper. The price to pay for this is twofold:
(i) We have to define Lefschetz numbers valued in some larger group as $K_0(A) \otimes \mathbf{C}$.
(ii) We have to deal with W*-algebras instead of general unital C*-algebras.
To obtain these results we have got a number of by-product facts on the theory of Hilbert W*- and C*-modules and on bounded module operators on them which are of independent interest.

The present paper is organized as follows: In §2 we prove the necessary facts on Hilbert W*-modules and their bounded module mappings extending results of W. L. Paschke [14], J.-F. Havet [5] and the first author [3]. In §3 we define Lefschetz numbers of two types and show the main properties of them. In §4 we discuss the C*-case and obstructions to refine the main results of §3.

Our standard references for the theory of Hilbert C*-modules are the papers [14, 15, 2, 9, 3, 10, 11] and the book of E. C. Lance [8]. The topological considerations are based on the publications [12, 13, 17, 18, 19, 11]. We are going to continue the investigations presented therein.

2 Hilbert W*-modules and module mappings

We want to show some more very nice properties of Hilbert W*-modules which often do not appear in the general C*-case. This partial class of Hilbert C*-modules was brought to the attention of the public by W. L. Paschke in his classical paper [14], and they are of use in many cases. The facts below can be reproved for the class of monotone complete C*-algebras carrying out much technical work, cf. [4], but not for larger classes of C*-algebras, in general. However, since we are going to understand the structure of general Hilbert C*-modules and their C*-duals better it suffices to treat the W*-case, and we can avoid these technicalities. Let us start with a property generalizing the (double) annihilator property of arbitrary subsets of W*-algebras.

Lemma 1 *Let A be a W*-algebra and $\{\mathcal{M}, \langle .,. \rangle\}$ be a Hilbert A-module. For every subset $\mathcal{S} \subseteq \mathcal{M}$ the bi-orthogonal set $\mathcal{S}^{\perp\perp} \subseteq \mathcal{M}$ is a Hilbert A-submodule and a direct summand of $\mathcal{M}$, as well as the orthogonal complement $\mathcal{S}^{\perp}$.*

Proof: The property of $\mathcal{S}^{\perp\perp} \subseteq \mathcal{M}$ to be a Hilbert A-submodule is obvious by the definition of orthogonal complements. Since the A-dual Banach A-module $\mathcal{M}'$ of $\mathcal{M}$ is a self-dual Hilbert A-module by [14, Th. 3.2] one can consider the Hilbert A-submodule $\mathcal{N}$ of $\mathcal{M}'$ consisting of the direct sum of $\mathcal{S}^{\perp\perp} \hookrightarrow \mathcal{M}'$ and of the Hilbert A-module of all A-linear bounded mappings from $\mathcal{M}$ to A vanishing on $\mathcal{S}^{\perp\perp}$. The second summand is the orthogonal complement of $\mathcal{S}^{\perp\perp}$ with respect to $\mathcal{M}'$ by construction and hence, it is a self-dual Hilbert A-submodule and direct summand of $\mathcal{N}$ by [3, Th. 3.2,Th. 2.8]. Consequently, the canonical embedding of $\mathcal{S}^{\perp\perp}$ into $\mathcal{N}$ is a direct summand of $\mathcal{N}$, and because of the submodule inclusion $\mathcal{S}^{\perp\perp} \subseteq \mathcal{M} \hookrightarrow \mathcal{N}$ it is a direct summand of $\mathcal{M}$, too. •

Example 1 below shows that situations different to that described at Lemma 1 can appear e. g. for Hilbert C*-modules over the C*-algebra $A = C([0,1])$.

Lemma 2 *Let A be a W*-algebra, $\{\mathcal{M}, \langle .,. \rangle\}$ be a Hilbert A-module and ϕ be a bounded module operator on it. Then the kernel $\mathrm{Ker}(\phi)$ of ϕ is a direct summand of $\mathcal{M}$ and has the property $\mathrm{Ker}(\phi) = \mathrm{Ker}(\phi)^{\perp\perp}$.*

Proof: By [14, Prop. 3.6] every bounded module operator ϕ on $\mathcal{M}$ continues to a bounded module operator on its A-dual Hilbert A-module $\mathcal{M}'$. The kernel of the extended operator is a direct summand of $\mathcal{M}'$ because of the completeness of its unit ball with respect to the τ_2-convergence induced by the functionals $\{f(\langle ., y \rangle) : f \in A_{*,1}, y \in \mathcal{M}'\}$ there, (cf. [3, Th. 3.2]). Consequently, the kernel of ϕ inside $\mathcal{M}$ has to coincide with its bi-orthogonal complement in $\mathcal{M}$, and by Lemma 1 it is a direct summand. •

Example 1 Note, that the kernel of bounded A-linear operators on Hilbert A-modules over arbitrary C*-algebras A is not a direct summand, in general. For example, consider the C*-algebra $A = C([0,1])$ of all continuous functions on the interval [0,1] as a Hilbert A-module over itself equipped with the standard inner product $\langle a, b\rangle_A = ab^*$. Define the mapping ϕ_g by the formula $\phi_g(f) = g \cdot f$ for a fixed function

$$g(x) = \begin{cases} -2x+1 & : \quad x \leq 1/2 \\ 0 & : \quad x \geq 1/2 \end{cases}$$

and for every $f \in A$. Then $\mathrm{Ker}(\phi_g)$ equals to the Hilbert A-submodule and (left) ideal $\{f \in A : f(x) = 0 \text{ for } x \in [0,1/2]\}$, being not a direct summand of A, but nevertheless, coinciding with the bi-orthogonal complement of itself with respect to A.

Corollary 1 *Let A be a W*-algebra, $\mathcal{M}$ and $\mathcal{N}$ be two Hilbert A-modules and $\phi : \mathcal{M} \to \mathcal{N}$ be a bounded A-linear mapping. Then the kernel* $\mathrm{Ker}(\phi)$ *of ϕ is a direct summand of $\mathcal{M}$ and has the property* $\mathrm{Ker}(\phi) = \mathrm{Ker}(\phi)^{\perp\perp}$.

Proof: Consider the Hilbert A-module $\mathcal{K}$ formed as the direct sum $\mathcal{K} = \mathcal{M} \oplus \mathcal{N}$ equipped with the A-valued inner product $\langle .,. \rangle_{\mathcal{M}} + \langle .,. \rangle_{\mathcal{N}}$. The mapping ϕ can be identified with a bounded A-linear mapping ϕ' on $\mathcal{K}$ acting on the direct summand $\mathcal{M}$ as ϕ and on the direct summand $\mathcal{N}$ as the zero operator. Since the kernel of ϕ' is a direct summand of $\mathcal{K}$ containing $\mathcal{N}$ by Lemma 2 its orthogonal complement is a direct summand of $\mathcal{M}$. The desired result turns out. •

Now we are in the position to give a description of the inner structure of arbitrary Hilbert W*-modules generalizing an analogous statement for self-dual Hilbert W*-modules by W. L. Paschke ([14, Th. 3.12]).

Proposition 1 *Let A be a W*-algebra and $\{\mathcal{M}, \langle .,. \rangle\}$ be a left Hilbert A-module. Then $\mathcal{M}$ is the closure of a direct orthogonal sum of a family $\{D_\alpha : \alpha \in I\}$ of norm-closed left ideals $D_\alpha \subseteq A$, where the closure of this direct sum is predetermined by the given on $\mathcal{M}$ A-valued inner product $\langle .,. \rangle$ and the A-valued inner products on the ideals are the standard A-valued inner product on A. Moreover, for every bounded A-linear mapping $r : \mathcal{M} \to A$ there is a net $\{x_\beta : \beta \in J\}$ of elements of $\mathcal{M}$ for which the limit*

$$\|.\|_A - \lim_{\beta \in J} \langle y, x_\beta \rangle$$

exists for every $y \in \mathcal{M}$ and equals $r(y)$.

Proof: Fix an arbitrary bounded A-linear mapping $r : \mathcal{M} \to A$. The kernel of r is a direct summand of $\mathcal{M}$ by Corollary 1. Consider its orthogonal complement. Since r can be continued to an bounded A-linear mapping $r(\cdot) = \langle ., x_r \rangle$ on the A-dual (self-dual) Hilbert A-module $\mathcal{M}'$ of $\mathcal{M}$ ($x_r \in \mathrm{Ker}(r)^\perp \subseteq \mathcal{M}'$) and since the orthogonal complement of the kernel of r inside $\mathcal{M}'$ is a direct summand isomorphic to $\{Ap, \langle ., . \rangle\}$ for some projection $p \in A$ by the structural theorem [14, Th. 3.12] for self-dual Hilbert W*-modules the orthogonal complement of the kernel of r with respect to $\mathcal{M}$ is isomorphic to the Hilbert A-module $\{I, \langle ., . \rangle_A\}$ for some norm-closed left ideal $I \subseteq Ap$ of A, where the left-strict closure of the left ideal I is the w*-closed ideal Ap of A. Now, r can be identified with the element $x_r \in Ap$, and $x_r \in Ap$ is the left-strict limit of a net $\{x_\beta : \beta \in J\}$ of elements of $I \cap \mathcal{M}$, cf. [16, §3.12].
Finally, by transfinit induction one has to decompose $\mathcal{M}$ into a sum of pairwise orthogonal direct summands of type $\mathrm{Ker}(r)^\perp$ for bounded A-linear functionals r on $\mathcal{M}$, where $\mathrm{Ker}(r)^\perp$ is always isomorphic to a left norm-closed ideal I of A with the standard A-valued inner product on it. •

We go on to investigate the image of bounded module mappings between Hilbert W*-modules. In general, many quite non-regular things can happen as the example below shows, but embeddings of self-dual Hilbert W*-modules into other Hilbert W*-modules can be shown to be mappings onto direct summands in contrast to the situation for general Hilbert C*-modules.

Example 2 Let A be the set of all bounded linear operators $B(H)$ on a separable Hilbert space H with basis $\{e_i : i \in \mathbf{N}\}$. Denote by k the operator $k(e_i) = \lambda_i e_i$ for a sequence $\{\lambda_i : i \in \mathbf{N}\}$ of non-zero positive real numbers converging to zero. Then the mapping

$$\phi_k : A \to A \quad , \quad \phi_k : a \to a \cdot k$$

is a bounded A-linear mapping on the left projective Hilbert A-module A. But the image is not a direct summand of this A-module and is not even Hilbert because direct summands of A are of the form Ap for some projection p of A, and $1_A \cdot k$ should equal p. The image of ϕ_k is a subset of the set of all compact operators on H. Note, that the mapping ϕ_k is not injective.

The following proposition was proved for arbitrary C*-algebras A, countably generated Hilbert A-modules $\mathcal{M}, \mathcal{N}$ without self-duality restriction and an injective bounded module mapping $\alpha : \mathcal{M} \to \mathcal{N}$ with norm-dense range by H. Lin [10, Th. 2.2]. We present another variant for a similar situation in the W*-case.

Proposition 2 *Let A be a W*-algebra, $\mathcal{M}$ be a self-dual Hilbert A-module and $\{\mathcal{N}, \langle ., . \rangle\}$ be another Hilbert A-module. Suppose, there exists an injective bounded module mapping*

$\alpha : \mathcal{M} \to \mathcal{N}$ with the range property $\alpha(\mathcal{M})^{\perp\perp} = \mathcal{N}$. Then the operator $\alpha(\alpha^*\alpha)^{-1/2}$ is a bounded module isomorphism of $\mathcal{M}$ and $\mathcal{N}$. In particular, they are isomorphic as Hilbert A-modules.

Proof: The mapping α possesses an adjoint bounded module mapping $\alpha^* : \mathcal{N} \to \mathcal{M}$ because of the self-duality of $\mathcal{M}$, cf. [14, Prop. 3.4]. Since $\alpha^*\alpha$ is a positive element of the C*-algebra $\mathrm{End}_A(\mathcal{M})$ of all bounded (adjointable) module mappings on the Hilbert A-module $\mathcal{M}$ the square root of it, $(\alpha^*\alpha)^{1/2}$, is well-defined by the series

$$(\alpha^*\alpha)^{1/2} = \|.\| - \lim_{n\to\infty} \|(\alpha^*\alpha)\|^{1/2} \left(\mathrm{id}_{\mathcal{M}} - \sum_{k=1}^{n} \lambda_k \left(\mathrm{id}_{\mathcal{M}} - \frac{(\alpha^*\alpha)}{\|(\alpha^*\alpha)\|} \right)^k \right)$$

with coefficients $\{\lambda_k\}$ taken from the Taylor series at zero of the complex-valued function $f(x) = \sqrt{1-x}$ on the interval [0,1]. Moreover, because

$$\langle (\alpha^*\alpha)^{1/2}(x), (\alpha^*\alpha)^{1/2}(x) \rangle = \langle \alpha(x), \alpha(x) \rangle$$

and because of the injectivity of α the mapping $(\alpha^*\alpha)^{1/2}$ has trivial kernel. At the contrary one can only say that the range of $(\alpha^*\alpha)^{1/2}$ is τ_1-dense in $\mathcal{M}$, (cf. [3]). Indeed, for every A-linear bounded functional $r(\cdot) = \langle ., y \rangle$ on the self-dual Hilbert A-module $\mathcal{M}$ mapping the range of $(\alpha^*\alpha)^{1/2}$ to zero one has

$$0 = \langle (\alpha^*\alpha)^{1/2}(x), y \rangle = \langle x, (\alpha^*\alpha)^{1/2}(y) \rangle$$

for every $x \in \mathcal{M}$. Hence, $y = 0$ since $(\alpha^*\alpha)^{1/2}$ is injective and $x \in \mathcal{M}$ was arbitrarily chosen.
Now, consider the mapping $\alpha(\alpha^*\alpha)^{-1/2}$ where it is defined on $\mathcal{M}$. Since $(\alpha^*\alpha)^{1/2}$ has both τ_1-dense range and trivial kernel by the assumptions on α its inverse unbounded module operator $(\alpha^*\alpha)^{-1/2}$ is τ_1-densely defined. One obtains

$$\langle \alpha(\alpha^*\alpha)^{-1/2}(x), \alpha(\alpha^*\alpha)^{-1/2}(y) \rangle = \langle x, y \rangle$$

for every x, y from the (τ_1-dense) area of definition of $(\alpha^*\alpha)^{-1/2}$. Consequently, the operator $\alpha(\alpha^*\alpha)^{-1/2}$ continues to a bounded isometric module operator on $\mathcal{M}$ by τ_1-continuity. The range of it is τ_1-closed (i.e., a self-dual direct summand of $\mathcal{N}$) and hence, equals $\mathcal{N}$ by assumption. Finally, since the range of $(\alpha^*\alpha)^{-1/2}$ is norm-closed and τ_1-dense in $\mathcal{M}$ and since $\mathcal{M}$ is self-dual the mapping α is a (non-isometric, in general) Hilbert A-module isomorphism itself. •

Corollary 2 *Let A be a W*-algebra, $\mathcal{M}$ be a self-dual Hilbert A-module and $\{\mathcal{N}, \langle ., . \rangle\}$ be another Hilbert A-module. Every injective module mapping from $\mathcal{M}$ into $\mathcal{N}$ is a Hilbert A-module isomorphism of $\mathcal{M}$ and of a direct summand of $\mathcal{N}$.*

For our application in §3 we need the following partial result:

Corollary 3 *Let A be a W*-algebra, $\mathcal{M}$ and $\mathcal{N}$ be countably generated Hilbert A-modules and $F : \mathcal{M} \to \mathcal{N}$ be a Fredholm operator (see [13]). Then* $\operatorname{Ker} F$ *and* $(\operatorname{Im} F)^\perp$ *are projective finitely generated A-submodules, and* $\operatorname{Ind} F = [\operatorname{Ker} F] - [(\operatorname{Im} F)^\perp]$ *inside $K_0(A)$.*

Proof: We denote by $\hat{\oplus}$ the direct orthogonal sum of two Hilbert A-modules, whereas $\oplus$ denotes the direct topological sum of two Hilbert A-submodules of a given Hilbert A-module, where orthogonality of the two components is not required. Let $\mathcal{M} = \mathcal{M}_0 \hat{\oplus} \mathcal{M}_1$, $\mathcal{N} = \mathcal{N}_0 \oplus \mathcal{N}_1$ be the decompositions from the definition of A-Fredholm operator:

$$F = \begin{pmatrix} F_0 & 0 \\ 0 & F_1 \end{pmatrix} : \mathcal{M}_0 \hat{\oplus} \mathcal{M}_1 \to \mathcal{N}_0 \oplus \mathcal{N}_1,$$

$F_0 : \mathcal{M}_0 \cong \mathcal{N}_0$, $F_1 : \mathcal{M}_1 \to \mathcal{N}_1$, $\mathcal{M}_1$ and $\mathcal{N}_1$ are the projective finitely generated modules. Let $x = x_0 + x_1$, $x_0 \in \mathcal{M}_0$ and $x_1 \in \mathcal{M}_1$, and $F(x) = 0$, so $0 = F_0(x_0) + F_1(x_1) \in \mathcal{N}_0 \oplus \mathcal{N}_1$. Thus $F_0(x_0) = 0$, $F_1(x_1) = 0$, so $x_0 = 0$ and $x \in \mathcal{M}_1$. Thus $\operatorname{Ker} F = \operatorname{Ker} F_1 \subset \mathcal{M}_1$. By Lemma 2 $\operatorname{Ker} F$ is a projective finitely generated A-module and has an orthogonal complement. So, by Corollary 2,

$$F = \begin{pmatrix} F_0 & 0 & 0 \\ 0 & F_1' & 0 \\ 0 & 0 & 0 \end{pmatrix} : \mathcal{M}_0 \hat{\oplus} \mathcal{M}_1' \oplus \operatorname{Ker} F \to \left(\mathcal{N}_0 \oplus \overline{F(\mathcal{M}_1')}\right) \hat{\oplus} (\operatorname{Im} F)^\perp$$

and $\operatorname{Ind} F = [\operatorname{Ker} F] - [(\operatorname{Im} F)^\perp]$. •

The following example shows that the situations may be quite different for general Hilbert C*-modules and injective mappings between them:

Example 3 Consider the C*-algebra $A = C([0,1])$ of all continuous functions on the interval [0,1] as a self-dual Hilbert A-module over itself equipped with the standard A-valued inner product $\langle a, b \rangle_A = ab^*$. The mapping $\phi : f(x) \to x \cdot f(x)$, $(x \in [0,1])$, is an injective bounded module mapping. Its range has trivial orthogonal complement, but it is not closed in norm and, consequently, not a direct summand of A. Nevertheless, the bi-orthogonal complement of the range of ϕ with respect to A equals A.

Lemma 3 *Let A be a W*-algebra. Let $\mathcal{P}$ and $\mathcal{Q}$ be self-dual Hilbert A-submodules of a Hilbert A-module $\mathcal{M}$. Then $\mathcal{P} \cap \mathcal{Q}$ is a self-dual Hilbert A-module and direct summand of $\mathcal{M}$. Moreover, $\mathcal{P} + \mathcal{Q} \subseteq \mathcal{M}$ is a self-dual Hilbert A-submodule.*
If $\mathcal{P}$ is projective and finitely generated then the intersection $\mathcal{P} \cap \mathcal{Q}$ is projective and finitely generated, too. If both $\mathcal{P}$ and $\mathcal{Q}$ are projective and finitely generated then the sum $\mathcal{P} + \mathcal{Q}$ is also.

Proof. Let $p : \mathcal{M} = \mathcal{P} \oplus \mathcal{P}^\perp \to \mathcal{P}^\perp$ be the canonical orthogonal projection existing by [3, Th. 2.8], (cf. [2] for the projective case). Let $p_Q = p : \mathcal{Q} \to \mathcal{P}^\perp$. Since $\mathcal{Q}$ is a self-dual Hilbert A-module p_Q admits an adjoint operator and $\mathrm{Ker} p_Q \subseteq \mathcal{Q}$ is a direct summand by Lemma 2. Consequently, it is a self-dual Hilbert A-submodule of $\mathcal{Q} \subseteq \mathcal{M}$. But $\mathrm{Ker} p_Q = \mathcal{P} \cap \mathcal{Q}$. To derive the second assertion one has to apply the fact again that every self-dual Hilbert A-submodule is a direct summand, cf. [3].
If $\mathcal{P}$ is projective and finitely generated then every direct summand of it is projective and finitely generated, what shows the additional remarks. •

3 Lefschetz numbers

Throughout this section A denotes a W*-algebra. This restriction enables us to apply the results of the previuos section being valid only in the W*-case, in general.
Let U be a unitary operator in the projective finitely generated Hilbert A-module $\mathcal{P}$. Then

$$U = \int_{S^1} e^{i\varphi} \, dP(\varphi), \tag{1}$$

where $P(\varphi)$ is the projection valued measure valued in the W*-algebra of all bounded (adjointable) module operators on $\mathcal{P}$, and the integral converges with respect to the norm. So we have a bounded and measurable function

$$L(\mathcal{P}, U) : \, S^1 \to K_0(A), \, \varphi \mapsto [dP(\varphi)], \tag{2}$$

This function is bounded in the sense that there exists a projection which is greater then all values with respect to the partial order in the space of projections. Let us denote the set of such functions by $K_0(A)_S$.
Let us note that the Lefschetz numbers for compact group action considered in [19] can be thought of as evaluated (for unitary representation) in the subspace of finitely valued (simple) functions:

$$\mathrm{Simple}(S^1, K_0(A)) \subset K_0(A)_S.$$

Suppose, $\mathcal{P} = A^n$. In the case of $L(\mathcal{P}, U) \in \mathrm{Simple}(S^1, K_0(A))$ associate with the integral (1)

$$\int_{S^1} e^{i\varphi} \, dP(\varphi) = \sum_k e^{i\varphi_k} P(\mathcal{E}_k)$$

the following class of the cyclic homology $HC_{2l}(M(n, A))$:

$$\sum_k P(\mathcal{E}_k) \otimes \ldots \otimes P(\mathcal{E}_k) \cdot e^{i\varphi_k}.$$

Passing to the limit we get the following element

$$\tilde{T}U = \int_{S^1} e^{i\varphi}\, d(P \otimes \ldots \otimes P)(\varphi) \in HC_{2l}(M(n, A)).$$

Then we define

$$T(U) = \mathrm{Tr}\,^n_*(\tilde{T}U) \;\in\; HC_{2l}(A),$$

where $\mathrm{Tr}\,^n_*$ is the trace in cyclic homology.

Lemma 4 ([19, Lemma 6.1])
Let $J : \mathcal{M} = A^m \to \mathcal{N} = A^n$ be an isomorphism, $U_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$, $U_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}$ be A-unitary operators and $JU_{\mathcal{M}} = U_{\mathcal{N}}J$. Then

$$T(U_{\mathcal{M}}) = T(U_{\mathcal{N}}).$$

Similar techniques can be developed for a projective finitely generated A-module $\mathcal{N}$ instead of A^n. For this purpose we take $\mathcal{N} = q(A^n)$, where q denotes the orthogonal projection from A^n onto its direct orthogonal summand $\mathcal{N}$. Then we set

$$U \oplus 1 : A^n \cong \mathcal{N} \oplus (1-q)A^n \to \mathcal{N} \oplus (1-q)A^n \cong A^n,$$

$$\tilde{T}U = \int_{S^1} e^{i\varphi}\, d(qPq \otimes \ldots \otimes qPq)(\varphi)\,.$$

The correctness is an immediate consequence of the Lemma 4.

Let us consider an A-elliptic complex (E, d) and its unitary endomorphism U. The results of §1 (cf. Prop. 2, Lemma 3, Lemma 3) and the standard Hodge theory argument help us to prove the following lemma.

Lemma 5 *For the A-Fredholm operator*

$$F = d + d^* : \Gamma(\mathcal{E}_{ev}) \to \Gamma(\mathcal{E}_{od}),$$

we have

$$\mathrm{Ker}\,(F|_{\Gamma(\mathcal{E}_{ev})}) \overset{\mathrm{def}}{=} H_{ev}(\mathcal{E}) = \;\oplus H_{2i}(\mathcal{E}),$$

$$\mathrm{Ker}\,(F|_{\Gamma(\mathcal{E}_{od})}) \overset{\mathrm{def}}{=} H_{od}(\mathcal{E}) = \;\oplus H_{2i+1}(\mathcal{E}),$$

where $H_m(\mathcal{E})$ is the orthogonal complement to $\mathrm{Im}\, d \subset \mathrm{Ker}\, d \subset \Gamma(\mathcal{E}_m)$ and $H_m(\mathcal{E})$ are projective U-invariant Hilbert A-modules.

Proof. For $u_{2i} \in \Gamma(E_{2i})$ while

$$(d+d^*)(u_0+u_2+u_4+\ldots)=0$$

we have

$$du_0+d^*u_2=0,\ du_2+d^*u_4=0,\ldots$$

Together with the equality

$$(du,d^*v)=(d^2u,v)=0$$

one obtains

$$du_0=0,\ du_2=0,\ldots \quad ;d^*u_2=0,\ d^*u_4=0,\ldots.$$

what implies $u_{2i} \in \operatorname{Ker}(d+d^*)$. On the other hand for $v_2 \in \operatorname{Im} d$, $v_2 = dv_1$ we have

$$(v_2,u_2)=(dv_1,u_2)=(v_1,d^*u_2)=0.$$

Thus $u_{2i} \in H_{2i}(E)$. Conversely, let $u = u_0+u_2+\ldots$, $u_{2i} \in H_{2i}(E)$, i.e. $du_{2i}=0$, $(i = 0,1,2,\ldots)$, and for any $v_{2i-1} \in E_{2i-1}$ we have

$$(dv_{2i-1},u_{2i})=0, \quad (v_{2i-1},d^*u_{2i})=0,$$

so $d^*u_{2i}=0$. Thus $u \in \operatorname{Ker}(d+d^*)$. The invariance and projectivity follow from the proved identification and Corollary 3. •

Definition 1 We define *the Lefschetz number* L_1 as

$$L_1(\mathcal{E},U)=\sum_i(-1)^i\,T(U|H_i(\mathcal{E}))\in K_0(A)_S.$$

Definition 2 We define *the Lefschetz number* L_{2l} as

$$L_{2l}(\mathcal{E},U)=\sum_i(-1)^i\,T(U|H_i(\mathcal{E}))\in HC_{2l}(A).$$

After all the following theorem is evident:

Theorem 1 *Let the Chern character* Ch *be defined as in* [1, 6, 7]. *Then*

$$L_{2l}(\mathcal{E},U)=\int_{S^1}(\operatorname{Ch}{}^0_{2l})_*(L_1(\mathcal{E},U))(\varphi)\,d\varphi.$$

Remark 1 In situations, when the endomorphism V of the elliptic C*-complex represents as an element of a represented there amenable group G acting on the C*-complex then the A-valued inner products can be chosen G-invariant, what gives us the unitarity of V (see [11]). However, there is another obstruction demanding new approaches which will be shown at Example 4 below.

4 Obstructions in the C*-case and related topics

The aim of this chapter is to show some obstructions arising in the general Hilbert C*-module theory for more general C*-algebras than W*-algebras which cause the made restriction of the investigations in section three. The results underline the outstanding properties of Hilbert W*-modules. To handle the general C*-case we often need a basic construction introduced by W. L. Paschke and H. Lin. It gives a link between the W*-case and the general C*-case.

Remark 2 (cf.[9, Def. 1.3], [14, §4])
Let $\{\mathcal{M}, \langle .,. \rangle\}$ be a left pre-Hilbert A-module over a fixed C*-algebra A. The algebraic tensor product $A^{**} \otimes \mathcal{M}$ becomes a left A^{**}-module defining the action of A^{**} on its elementary tensors by the formula $ab \otimes h = a(b \otimes h)$ for $a, b \in A^{**}$, $h \in \mathcal{M}$. Now , setting

$$\left[\sum_i a_i \otimes h_i, \sum_j b_j \otimes g_j \right] = \sum_{i,j} a_i \langle h_i, g_j \rangle b_j$$

on finite sums of elementary tensors one obtains a degenerate A^{**}-valued inner pre-product. Factorizing $A^{**} \otimes \mathcal{M}$ by $N = \{z \in A^{**} \otimes \mathcal{M} : [z,z] = 0\}$ one obtains a pre-Hilbert A^{**}-module denoted by $\mathcal{M}^{\#}$ in the sequel. It contains $\mathcal{M}$ as a A-submodule. If $\mathcal{M}$ is Hilbert then $\mathcal{M}^{\#}$ is Hilbert, and vice versa. The transfer of the self-duality is more difficult. If $\mathcal{M}$ is self-dual then $\mathcal{M}^{\#}$ is self-dual, too. But,

> **Problem.** Suppose, the underlying C*-algebra A is unital. Whether the property of $\mathcal{M}^{\#}$ to be self-dual implies that $\mathcal{M}$ was already self-dual?

Other standard properties like e.g. C*-reflexivity can not be transferred. But every bounded A-linear operator T on $\mathcal{M}$ has a unique extension to a bounded A^{**}-linear operator on $\mathcal{M}^{\#}$ preserving the operator norm, (cf. [9, Def. 1.3]).

Proposition 3 *Let A be a C*-algebra, $\mathcal{M}$ and $\mathcal{N}$ be two Hilbert A-modules and $\phi : \mathcal{M} \to \mathcal{N}$ be a bounded A-linear mapping. Then the kernel* $\mathrm{Ker}(\phi)$ *of ϕ coincides with its bi-orthogonal complement inside $\mathcal{M}$. In general, it is not a direct summand.*

Proof: Let us assume, $\mathrm{Ker}(\phi) \neq \mathrm{Ker}(\phi)^{\perp\perp}$ with respect to the A-valued inner product of $\mathcal{M}$. Form the direct sum $\mathcal{L} = \mathcal{M} \oplus \mathcal{N}$. The mapping ϕ extends to a bounded A-linear mapping ψ on $\mathcal{L}$ setting

$$\psi(x) = \begin{cases} \phi(x) & : \quad x \in \mathcal{M} \\ 0 & : \quad x \in \mathcal{N} \end{cases} .$$

Extend ψ further to a bounded A^{**}-linear operator on the correspondent Hilbert A^{**}-module $\mathcal{L}^{\#}$. By Lemma 2 the sets $\mathrm{Ker}(\phi)^{\#}$ and $(\mathrm{Ker}(\phi)^{\perp\perp})^{\#}$ both are contained in the kernel $\mathrm{Ker}(\psi)$ of ψ, which is a direct summand of $\mathcal{L}^{\#}$ and fulfils $\mathrm{Ker}(\psi) = \mathrm{Ker}(\psi)^{\perp\perp}$. This contradicts the assumption.
The second assertion follows from Example 1. •

Corollary 4 *Let A be a C*-algebra and $\{\mathcal{M}, \langle .,. \rangle\}$ be a Hilbert A-module. The kernel* $\mathrm{Ker}(r)$ *of every bounded module mapping $r : \mathcal{M} \to A$ coincides with its bi-orthogonal complement inside $\mathcal{M}$, but it is not a direct summand, in general.*

Corollary 5 *Let A be a C*-algebra and $\{\mathcal{M}, \langle .,. \rangle\}$ be a Hilbert A-module. Suppose, there exists a bounded module mapping $r : \mathcal{M} \to A$ with the property* $\mathrm{Ker}(r)^{\perp} = \{0\}$*. Then r is the zero mapping.*

Lemma 6 *Let A be a C*-algebra and $\{\mathcal{M}, \langle .,. \rangle\}$ be a (left) Hilbert A-module. For every bounded module mapping $r : \mathcal{M} \to A$ the subset* $\mathrm{Ker}(r)^{\perp} \subseteq \mathcal{M}$ *is a Hilbert A-submodule, and it is isomorphic as a Hilbert A-module to a norm-closed (left) ideal D of A equipped with the standard A-valued inner product $\langle .,. \rangle_A$.*

Proof: By Corollary 4 the set $\mathrm{Ker}(r)^{\perp} \subseteq \mathcal{M}$ can be assumed to be non-zero, in general. Again, form the Hilbert A^{**}-module $\mathcal{M}^{\#}$ and extend r to a bounded A^{**}-linear mapping r' on it. The kernel of r' is a direct summand of $\mathcal{M}^{\#}$ isomorphic to a (left) norm-closed ideal of A^{**} as a Hilbert A^{**}-module by Corollary 1 and Proposition 1. Consequently, $\mathrm{Ker}(r) \subseteq \mathrm{Ker}(r') \cap \mathcal{M} \subseteq \mathcal{M}^{\#}$ is isomorphic to a (left) norm-closed ideal D of A as a (left) Hilbert A-module. •

We want to get a structure theorem on the interrelation of Hilbert C*-modules and their C*-dual Banach C*-modules. To obtain the full picture define a new topology on (left) Hilbert C*-modules in analogy to the (right) strict topology on C*-algebras A:

Definition 3 Let A be a C*-algebra and $\{\mathcal{M}, \langle .,. \rangle\}$ be a (left) Hilbert A-module. A norm-bounded net $\{x_\alpha : \alpha \in I\}$ of elements of $\mathcal{M}$ is *fundamental with respect to the right $*$-strict topology* if and only if the net $\{\langle y, x_\alpha \rangle : \alpha \in I\}$ is a Cauchy net with respect to the norm topology on A for every $y \in \mathcal{M}$. The net $\{x_\alpha : \alpha \in I\}$ *converges to an element $x \in \mathcal{M}$ with respect to the right $*$-strict topology* if and only if

$$\lim_{\alpha \in I} \|\langle y, x - x_\alpha \rangle\|_A = 0$$

for every $y \in \mathcal{M}$.

Theorem 2 *Let A be a C*-algebra and $\{\mathcal{M}, \langle .,. \rangle\}$ be a (left) Hilbert A-module. The following conditions are equivalent:*

(i) *$\mathcal{M}$ is self-dual.*

(ii) *The unit ball of $\mathcal{M}$ is complete with respect to the right $*$-strict topology.*

Moreover, the linear hull of the completed with respect to the right $*$-strict topology unit ball of $\mathcal{M}$ coincides with the A-dual Banach A-module $\mathcal{M}'$ of $\mathcal{M}$.

Proof: First, let us show the equivalence (i)$\leftrightarrow$(ii). Suppose the unit ball of $\mathcal{M}$ is complete with respect to the right $*$-strict topology. Consider an arbitrary non-trivial bounded module mapping $r : \mathcal{M} \to A$ of norm one. Restrict the attention to the non-zero Hilbert A-submodule $\mathrm{Ker}(r)^\perp \subseteq \mathcal{M}$ being isomorphic as a Hilbert A-module to a norm-closed (left) ideal D of A equipped with the standard A-valued inner product $\langle .,. \rangle_A$ by Lemma 6. By [3, Th. 3.2] there exist nets $\{x_\alpha : \alpha \in I\} \subset \mathrm{Ker}(r)^\perp$ bounded in norm by one such that $\tau_2 - \lim_{\alpha \in I} x_\alpha = r$ inside the self-dual Hilbert A^{**}-module $((\mathrm{Ker}(r)^\perp)^\#)'$. But, the values $r(y)$, $y \in \mathrm{Ker}(r)^\perp$, all belong to A and, in particular, to the set of all right multipliers of the C*-subalgebra and two-sided ideal $B = \langle \mathrm{Ker}(r)^\perp, \mathrm{Ker}(r)^\perp \rangle$ of A. Therefore, there exists a special net $\{x_\alpha : \alpha \in I\} \subset \mathrm{Ker}(r)^\perp$ such that

$$\|.\|_{\mathcal{M}} - \lim_{\alpha \in I} b(\langle y, x_\alpha \rangle - r(y)) = 0$$

for every $y \in A$, every $b \in B$, cf. [16, §3.12]. Since the set $\{by : b \in B, y \in \mathrm{Ker}(r)^\perp\}$ is norm-dense in $\mathrm{Ker}(r)^\perp$ one implication is shown. The opposit one follows from the formula

$$r(y) = \|.\|_A - \lim_{\alpha \in I} \langle y, x_\alpha \rangle \, , \;\; y \in \mathcal{M} \, ,$$

defining a bounded module mapping $r : \mathcal{M} \to A$ for every norm-bounded fundamental with respect to the right $*$-strict topology net $\{x_\alpha : \alpha \in I\} \in \mathcal{M}$. By the way one has proved the conclusion that the A-dual Banach A-module $\mathcal{M}'$ of every Hilbert A-module $\mathcal{M}$ arises as the linear hull of the completed with respect to the right $*$-strict topology unit ball of $\mathcal{M}$. •

Corollary 6 *Let A be a C*-algebra and D be a norm-closed (left) ideal of A. Then $\{D, \langle .,. \rangle_A\}$ is self-dual if and only if there is a projection $p \in A$ such that $D \equiv Ap$ and $p \in D$.*

Proof: If D is self-dual then the identical embedding of D into A is a bounded A-linear mapping. It must be represented by an element $p \in D$ with the property $dp^* = d$ for every $d \in D$. That is, $pp^* = p \in D$ is positive and idempotent. The functional property of the mapping p gives the structure of D as $D \equiv Ap$. •

Theorem 3 *Let A be a C*-algebra and $\{\mathcal{M}, \langle .,. \rangle\}$ be a (left) Hilbert A-module. The following conditions are equivalent:*

1. *$\mathcal{M}$ is A-reflexive.*

2. *Every norm bounded net $\{x_\alpha : \alpha \in I\}$ of elements of $\mathcal{M}$ for which all the nets $\{r(x_\alpha) : \alpha \in I\}$, $(r \in \mathcal{M}')$, converge with respect to $\|.\|_A$ has its limit x inside $\mathcal{M}$.*

Moreover, the linear hull of the completed with respect to this topology unit ball of $\mathcal{M}$ coincides with the A-bidual Banach A-module $\mathcal{M}''$ of $\mathcal{M}$.

Proof: Suppose $\mathcal{M}$ is not self-dual because otherwise one simply refers to Theorem 2. Obviously, the linear hull of the completion of the unit ball of $\mathcal{M}$ with respect to this topology is a Banach A-module $\mathcal{N}$. Continue the A-valued inner product from $\mathcal{M}$ to $\mathcal{N}$ by the rule

$$\langle x, y \rangle = \lim_{\alpha \in I} \langle x_\alpha, y \rangle$$

for every element $\langle ., y \rangle \in \mathcal{M}'$, where $y \in \mathcal{M}$. Since the net converges with respect to the right $*$-strict topology on $\mathcal{M}$, too, the limit x can be interpreted as an A-linear bounded functional on $\mathcal{M}$. This lets to the definition of the value $\langle x, x \rangle$ in the same manner. Consequently, $\mathcal{N}$ is a Hilbert A-module containing $\mathcal{M}$ as a Hilbert A-submodule and possessing the same A-dual Banach A-module $\mathcal{M}' \equiv \mathcal{N}'$. (Cf. [15] for similar constructions.) Moreover, the unit ball of $\mathcal{N}$ is complete with respect to the new topology. Since the A-valued inner product on $\mathcal{M}$ can be continued to an A-valued inner product on $\mathcal{M}'' \equiv \mathcal{N}''$ by [15, Th. 2.4] every element of $\mathcal{M}''$ can be described in this way, and $\mathcal{N}$ is A-reflexive. •

Example 4 Consider the C*-algebra $A = C([0,1])$ of all continuous functions on the unit interval as a Hilbert A-module over itself. Let U be defined as

$$U(f)(t) = e^{it} f(t) \, , \; t \in [0,1] \, ,$$

a unitary operator. Take this unitary operator as the generator of a unitary representation of the amenable abelian group $\mathbf{Z}$. All complex irreducible representations of $\mathbf{Z}$ are one-dimensional. If we would like to apply A. S. Mishchenko's theorem in this case then we would have to have a finite spectrum for the generator U of the representation what is not the case. Beside this, the only projections inside A and, therefore, the only self-adjoint idempotent module operators on A are 1_A and 0_A, and there exists no spectral decomposition of elements and no non-trivial direct A-module summand inside A.

Remark 3 As it is known in all sufficient cases the morphism S gives an isomorphism of $HC_{2l}(A)$ and $HC_0(A)$ and we can work only with the second group. In this situation we can define the Lefschetz number $L_0 \in HC_0(A)$ as in [18] for general C*-algebras A.
But for K-groups valued numbers even in the case of an action of an e.g. amenable group G (see Example 4) we need some kind of infinitness and convergence, so we have to pass to $K_0(A)_S$. The natural expression of this infinitness of eigenvalues is the spectral decomposition, so we have to work with W*-algebras, at least for L_1. The crucial moment is that in this situation there is no theorem like [12].
Surely this argument is quite unexplicite and we have a chance for refinement e.g. for the monotone complete C*-algebras. But, the techniques for the monotone complete case are rather complicated and the results do only differ slightly from that of the W*-case, cf. [4].

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